Gradient Descent

Statistical Methods in NLP 2
ISCL-BA-08

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Summer Semester 2025

version: 3a5a03e+ @2025-05-16

Optimization and machine learning

- Most machine learning problems are optimization problems:
  - define a *model* (a function) class (parametrized by a set of parameters w)
  - define an *objective* (or *loss*, *cost*) function $J(w)$
  - find the weights (model) that minimize the objective function

$$w^* = \arg\max_{w} J(w)$$

- If the objective function is continuous and differentiable we can use derivative of the function to find the minimum
- If the loss function is *convex*, there is a single *global minimum*
- If we are lucky (e.g., as in regression), there is an analytic solution to $J(w) = 0$
- In most cases we use a search procedure to find the minimum

Estimating regression parameters (again)

analytic solution

- Data: $\boldsymbol{x} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \quad \boldsymbol{y} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

  Model: $\hat{y} = wx$

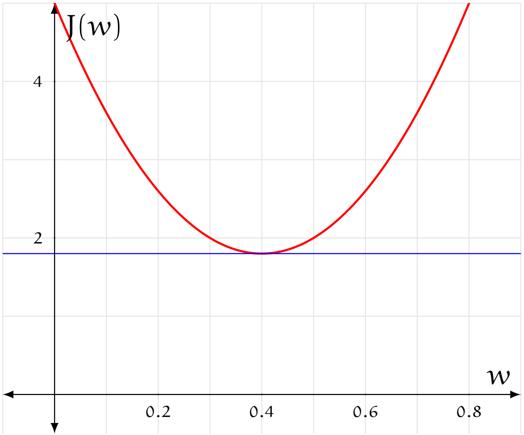
Estimating regression parameters (again)

analytic solution

- Data: $\mathbf{x} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

  Model: $\hat{y} = wx$
- Squared errors

$$
\begin{aligned}
J(w) &= (4w - 1)^2 + (2w - 2)^2 \\
&= 20w^2 - 16w + 5
\end{aligned}
$$



Estimating regression parameters (again)

analytic solution

- Data: $\mathbf{x} = \begin{bmatrix}4\\2\end{bmatrix} \quad \mathbf{y} = \begin{bmatrix}1\\2\end{bmatrix}$
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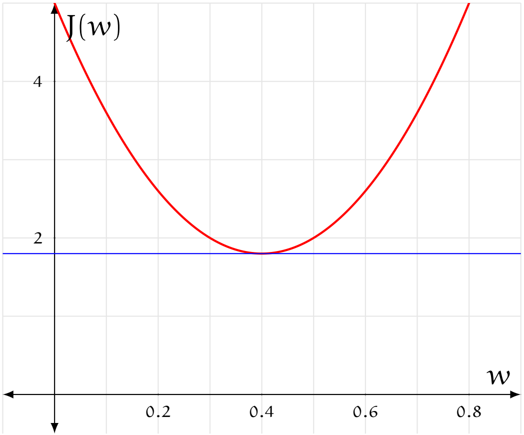
- Squared errors

$$\begin{aligned} J(w) &= (4w-1)^2 + (2w-2)^2 \\ &= 20w^2 - 16w + 5 \end{aligned}$$

- Setting the derivative to zero:

$$\frac{dJ}{dw} = 40w - 16 = 0 \Rightarrow w = \frac{2}{5}$$



Gradient descent for parameter estimation

- In many ML problems, we do not have a closed form solution for finding the minimum of the error function
- In these cases, we use a search strategy
- *Gradient descent* is a search method for finding a minimum of a (error) function
- The general idea is to approach a minimum of the error function in small steps

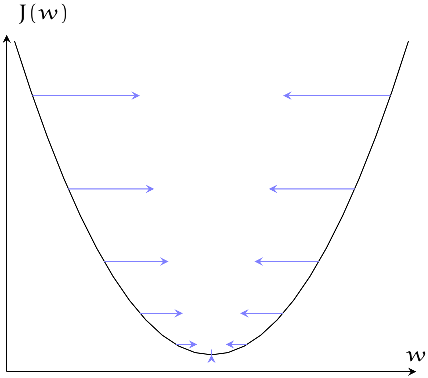
$$\boldsymbol{w} \leftarrow w - \eta \nabla J(\boldsymbol{w})$$

∇J is the gradient of the loss function, it points to the direction of the maximum increase

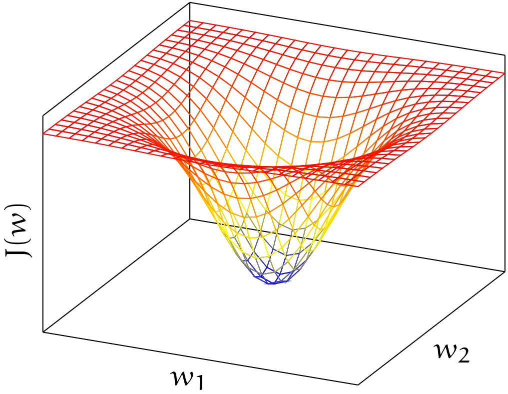
η is the *learning rate* or *step size*)

Gradient descent with single parameter

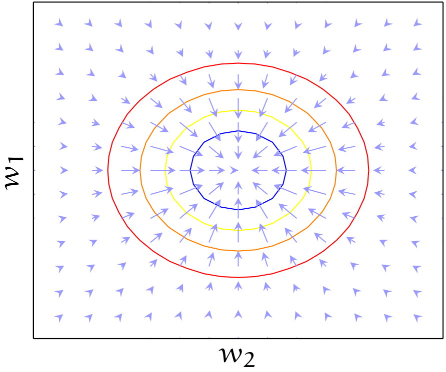
- For a single parameter, gradient is a one-dimensional vector
- The direction of gradient is towards the maximum increase
- We take steps proportional to $-\nabla J(w)$
- Steeper the curve, the larger the parameter update



Gradient descent with two/more parameters



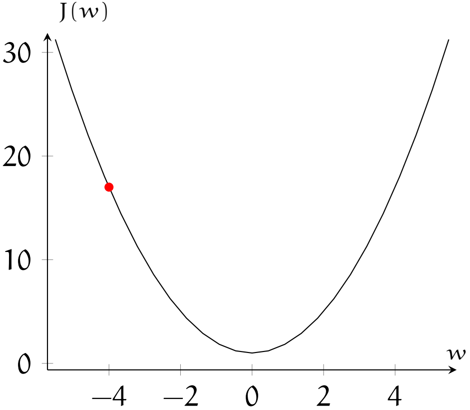
Objective function



Negative gradients

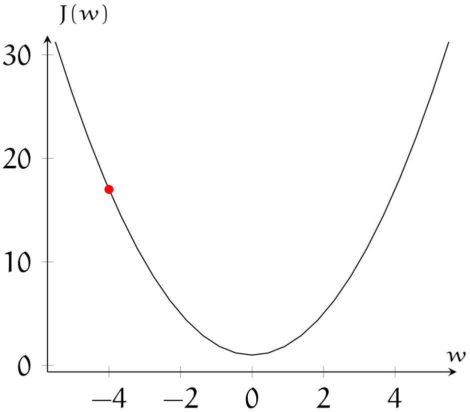
Gradient descent: demonstration

- $J(w) = x^2 + 1, J'(w) =$



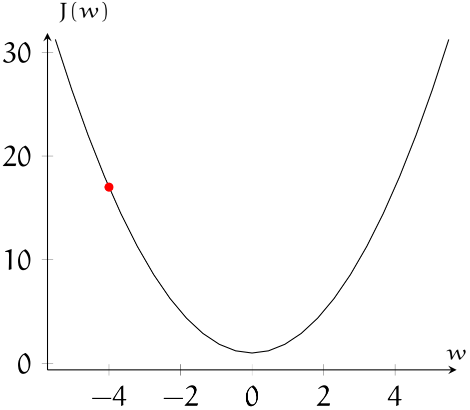
Gradient descent: demonstration

- $J(w) = x^2 + 1, J'(w) = 2x$



Gradient descent: demonstration

- $J(w) = x^2 + 1$, $J'(w) = 2x$
- Initialize $w = -4$, $\eta = 0.25$:
  $-0.25J'(-4) =$



Gradient descent: demonstration

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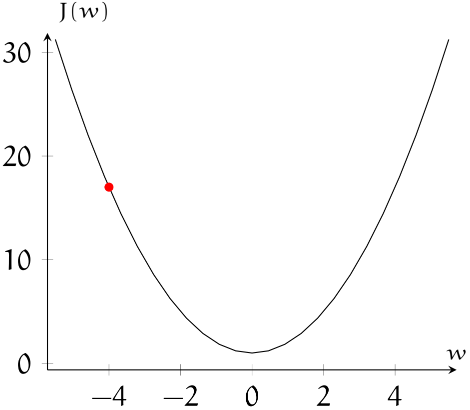

Gradient descent: demonstration

- $J(w) = x^2 + 1$, $J'(w) = 2x$
- Initialize $w = -4, \eta = 0.25$:
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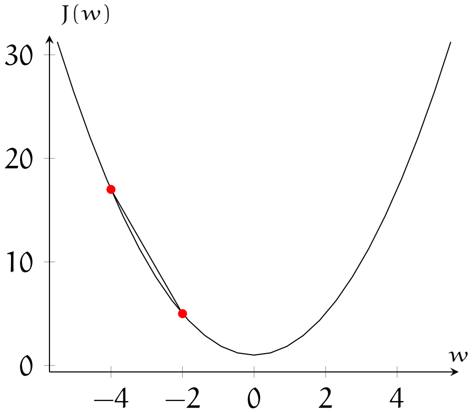
- $w = -4 + 2 = -2$:
  $-0.25J'(-2) = 1$



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- $w = -4 + 2 = -2$:
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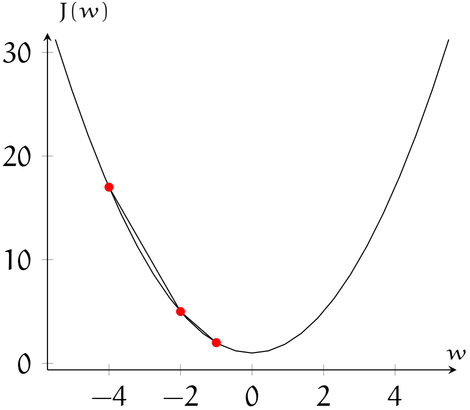
- $w = -2 + 1 = -1$:
  $-0.25J'(-1) = 0.5$



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- $w = -2 + 1 = -1$:
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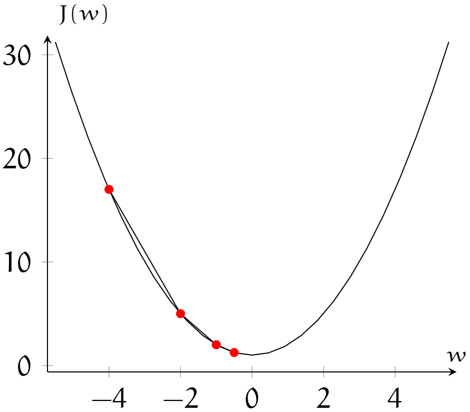
- $w = -1 + 0.5 = -0.5$:
  $-0.25J'(-0.5) = 0.25$



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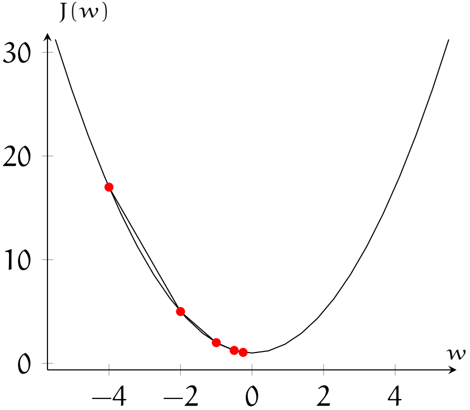
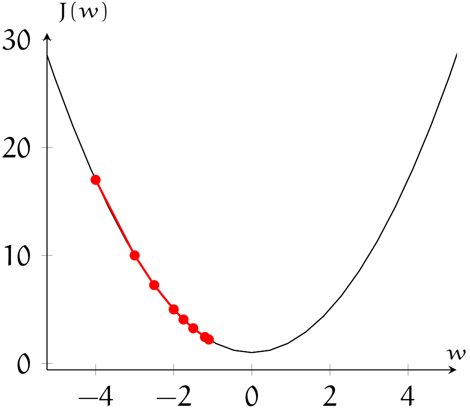
  $-0.25J'(-1) = 0.5$
- $w = -1 + 0.5 = -0.5$:
  $-0.25J'(-0.5) = 0.25$
- ...when do we stop?



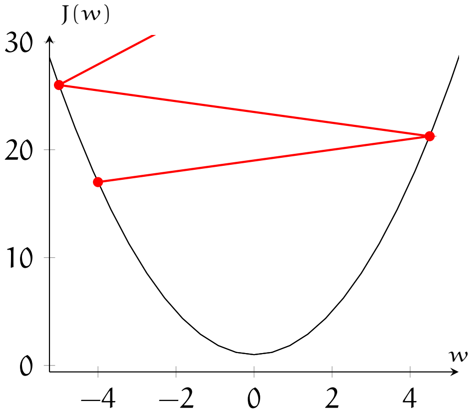
The importance of the learning rate

- Small learning rate causes slow convergence



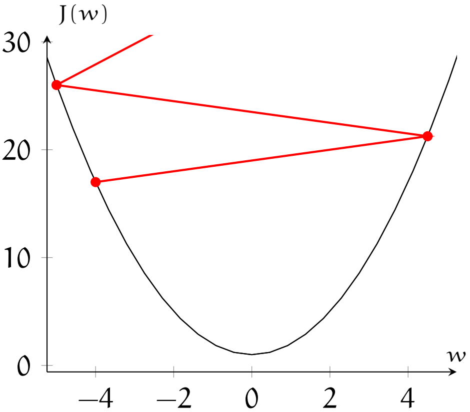
The importance of the learning rate

- Small learning rate causes slow convergence
- Too large learning rate causes ovrshooting



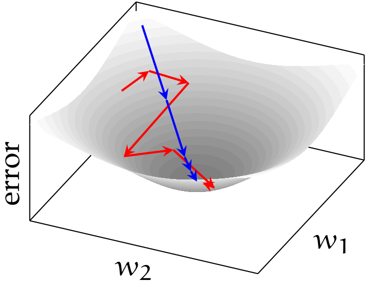
The importance of the learning rate

- Small learning rate causes slow convergence
- Too large learning rate causes ovrshooting
- It is common to adapt the learning rate



Stochastic gradient descent

- Standard (batch) gradient descent is computationally expensive: it updates weight at every *epoch*
- Stochastic gradient descent (SGD) updates weights for every training instance
- SGD may take more steps, but converges to the same solution



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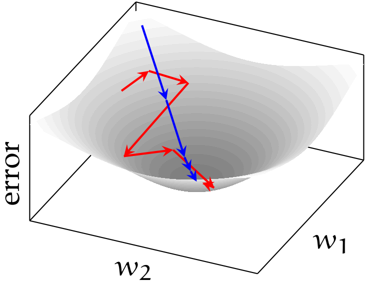

- In practice a *mini-batch* is more common
- Correct *batch size* is an important hyperparameter for neural networks

Adapting learning rate

- The choice of learning rate η is important

too small slow convergence
too big overshooting - may fluctuate around the minimum, or even jump away

- The idea is to adapt the learning rate during learning
- A common trick is adding a momentum:
  if we move in the same direction a long time accelerate

$$\Delta w_{ij}(t) = \eta \frac{\partial J}{\partial w_{ij}} + \alpha \Delta w_{ij}(t-1)$$

- Other improvements include using a separate learning rate for each parameter
- There are many adaptive optimization algorithms:
  Adagrad, Adadelta, RMSprop, Adam, ...

Summary

- Gradient descent is a general method for searching for the minima of a function
- We often use a mini-batch gradient descent: weights are based on a small number of training instances
- Reading: Jurafsky and Martin (2025, Section 5.6)

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Next:

- Introduction to ANNs, Reading: Jurafsky and Martin (2025, Chapter 7)

References & further reading

Jurafsky, Daniel and James H. Martin (2025). *Speech and Language Processing: An Introduction to Natural Language Processing, Computational Linguistics, and Speech Recognition with Language Models*. 3rd. Online manuscript released January 12, 2025. URL: `https://web.stanford.edu/~jurafsky/slp3/`.